Math 142 Lecture 14 Notes

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1 Fundamental Groups of Product and Orbit Spaces

1.1 Fundamental groups of product spaces

Last time, we stated the following theorem.

Theorem 1.1. *If X, Y are topological spaces, then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. We have continuous maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Define $\psi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ as

$$[\alpha] \mapsto ((p_1)_*([\alpha]), (p_2)_*([\alpha])) = ([p_1 \circ \alpha], [p_2 \circ \alpha]).$$

Injectivity: If $p_1 \circ \alpha \simeq_F e_{x_0}$ rel $\{0, 1\}$ (where e_{x_0} is the constant path at x_0) and $p_1 \circ \alpha \simeq_F e_{x_0}$ rel $\{0, 1\}$, then $\alpha \simeq_{(F,G)} e_{(x_0,y_0)}$ rel $\{0, 1\}$. So if $\psi([\alpha]) = (e, e)$, then $[\alpha] = e$. So ψ is injective.

Surjectivity: If $[\beta] \in \pi_1(X, x_0)$ and $[\gamma \in \pi_1(Y, y_0)$, let $\alpha : [0, 1] \to X \times Y$ be $\alpha(t) = (\beta(t), \gamma(t))$ for $t \in [0, 1]$. Then $\psi([\alpha]) = ([\beta], [\gamma])$. Hence, ψ is surjective, so ψ is an isomorphism. □

1.2 Orbit spaces

1.2.1 Definitions and examples of orbit spaces

Let G be a group. (G can be thought of as a topological group with the discrete topology)

Definition 1.1. A group G *acts* on a space X if for all $g \in G$, g defines a homeomorphism $f_g : X \to X$ such that

1. For the identity $e \in G$, $f_e = \mathrm{id}_X$.

2. $\forall g, h \in G$, $f_{gh} = f_h \circ f_g$.

G acts *properly discontinuously* (called "niecly") on X if G acts on X, and $\forall x \in X$ and $g \in G$ with $g \neq e$, there exists an open neighborhood U of x such that $U \cap f_g(U) = \varnothing$.

The "nice" condition implies that if $g \neq e$, then $f_g(x) \neq x$ for each $x \in X$; i.e. there are no fixed points.

Definition 1.2. Define an identification space X/G by choosing a partition $\mathcal{P}$ on X such that x, y are in the same subset in $\mathcal{P}$ iff there exists some $g \in G$ such that $f_g(x) = y$. This identification space is called an *orbit space*.

Example 1.1. Let $X = \mathbb{R}$, and let $\mathbb{Z}$ act on $\mathbb{R}$ by $f_n(x) = x+n$. The orbit space $\mathbb{R}/\mathbb{Z} \cong S^1$, with the homeomorphism $[x] \mapsto e^{2\pi i x}$.

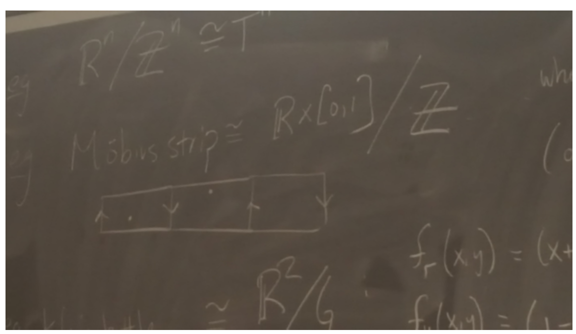
Example 1.2. Let $X = \mathbb{R}^2$, and let $\mathbb{Z}^2$ act on $\mathbb{R}$ by $f_{(m,n)}(x, y) = (x + m, y + n)$. The orbit space $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$, the torus.

This is because every $(x, y) \in \mathbb{R}^2$ is in the same equivalence class in the partition as some $(x', y') \in [0, 1] \times [0, 1]$. If (x, y) is in the box bounded by $x = m$, $x = m + 1$, $y = n$, and $y = n + 1$, then $(x' + y') = f_{(-m,-n)}(x, y)$ is in the desired unit square.

If we look at $[0, 1] \times [0, 1]$, the top and bottom edges get identified together by $f_{(0,1)}$, and the left and right edges get identified together by $f_{(1,0)}$. Nothing else gets identified (check this yourself), so we do indeed get the torus T^2.

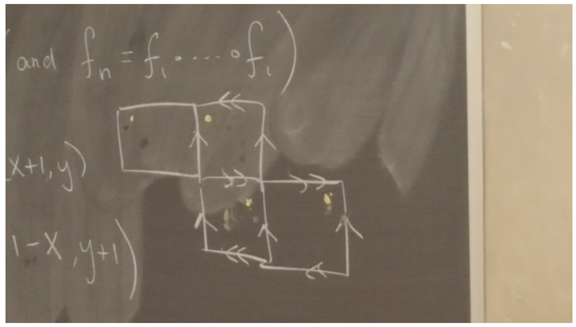
Example 1.3. More generally, $\mathbb{R}^n/\mathbb{Z}^n \cong T^n$. Morally, this is because the action of $\mathbb{Z}^n$ is the product of n actions, each acting on one component of $\mathbb{R}^n$

Example 1.4. The Möbius strip is homeomorphic to $(\mathbb{R} \times [0, 1])/\mathbb{Z}$, where the action is $f_1(x, y) = (x + 1, 1 - y)$ (and $f_n = f_1 \circ \cdots \circ f_1$ n times).



Example 1.5. The Klein bottle is homeomorphic to $\mathbb{R}^2/G$, where $G = \langle r, u \mid rur = u \rangle$, and the action is $f_r(x, y) = (x + 1, y)$, and $f_u(x, y) = (1 - x, y + 1)$. The group elements r

and u mean moving over right one square or up on square.



Example 1.6. Projective space $\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$, where $f_1(x) = -x$.

Example 1.7. The *Lens space*[1] $L(p,q)$ for p,q relatively prime and $p > q \geq 1$ is $S^3/(\mathbb{Z}/p\mathbb{Z})$, where we think of S^3 as the unit sphere in $\mathbb{R}^4 = \mathbb{C}^2$, and $f_1(z_1, z_2) = (e^{i2\pi/p} z_1, e^{i2\pi q/p} z_2)$.

Note that $e^{2\pi i/2} = -1$, so $L(2,1) \cong \mathbb{R}P^2$, so this generalizes projective space in some sense.

1.2.2 Fundamental groups of orbit spaces

Recall that simply connected means that $\pi_1(X) \cong 1$. Orbit spaces constructed from simply connected spaces have a lot of structure.

Theorem 1.2. *If G acts properly discontinuously (or "nicely") on a space X, and X is simply connected and path-connected, then $\pi_1(X/G) \cong G$.*

Proof. Let $p \in X$, and let $\pi : X \to X/G$ be the projection map (from the definition of the identification space). Let $q = \pi(p)$. If $\gamma : [0,1] \to X$ is a path from p to $f_g(p)$ (for some $g \in G$), then $(\pi \circ \gamma)(1) = \pi(\gamma(1)) = \pi(f_g(p)) = \pi(p) = q$. So $[\pi \circ \gamma] \in \pi_1(X/G, q)$.

X is simply connected, so any two such paths γ, γ' are homotopic rel $\{0,1\}$. So all we care about from γ is $\gamma(0)$ and $\gamma(1)$. Then define $\phi : G \to \pi_1(X/G, q)$ sending $g \mapsto [\pi \circ \gamma_g]$, where γ_g is a path in X from p to $f_g(p)$.

ϕ is a homomorphism: This is proved exactly like in the case $\mathbb{R} \to S^1$.

ϕ is surjective and injective: This is just like $\mathbb{R} \to S^1$, but let's give a little more description. Use:

1. Path lifting lemma: If σ is a path in X/G with $\sigma(0) = q$, there exists a unique path $\tilde{\sigma}$ in X such that $\tilde{\sigma}(0) = p$ and $\pi \circ \tilde{\sigma} = \sigma$.

2. Homotopy lifting lemma: If F is a homotopy rel $\{0,1\}$ of paths σ, σ' in X/G from q to q, then there exists a unique homotopy $\tilde{F}$ in X from the lifts $\tilde{\sigma}$ to $\tilde{\sigma}'$ (coming from path lifting) such that $\pi \circ \tilde{F} = F$.

[1] Professor Conway thinks about these in his research.

The truth of these lemmas follows from the fact that the action is "nice." $\square$

Corollary 1.1. *$\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.*

Corollary 1.2. *$\pi_1(\text{Möbius strip}) \cong \mathbb{Z}$.*

Corollary 1.3. *$\pi_1(\text{Klein bottle}) \cong \langle r, u \mid rur = u \rangle$.*

Corollary 1.4. *$\pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$.*